Comparisons of Signals

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March 2024

Abstract

A (Blackwell) experiment specifies the joint distribution of truth and the data generated by the experiment. A signal specifies the joint distribution of truth, the data generated by the signal, and the data generated by any other signal. Describing two experiments does not determine their joint informational content; describing two signals does. Blackwell (1953) studied (equivalent) comparisons of experiments; he characterized when one experiment is more valuable than another regardless of the preferences of the agent. We study (various, non-equivalent) comparisons of signals. Among other comparisons, we characterize when one signal is more valuable than another regardless of the preferences of the agent and regardless of what other information the agent may have. We show this comparison is equivalent to a new condition, termed *reveal-or-refine*, which says that for every piece of data that could be generated by the more valuable signal, either that data reveals the truth, or it refines the data generated by the less valuable signal.

JEL classification: C70; D82; D83; D85
Keywords: Experiments, signals, Blackwell order, sufficiency

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1 Introduction

Economic theory has long been concerned with comparing the value of information sources. Blackwell (1953), for instance, gives conditions under which one source is more valuable than another source, regardless of preferences. Most prior work, however, considers the value of an information source in isolation, in absence of other – potentially correlated – information sources. Concretely, suppose we wish to judge whether a subscription to the *New York Times* (*NYT*) is more valuable than a subscription to the *Washington Post* (*WP*),[1] regardless of the reader's interests. Blackwell's analysis tells us how to make this judgment, namely by comparing the distributions of beliefs induced by reading the *NYT* vs. the *WP*. However, a conclusion based on this procedure might be mistaken if the reader already has an existing subscription to a newspaper. Most obviously, if the reader already subscribes to the *NYT*, a subscription to the *WP* is likely more valuable than a duplicate subscription to the *NYT*. More subtly, a subscription to a third newspaper such as the *Wall Street Journal* (*WSJ*) might flip the comparison and make the *WP* more valuable than the *NYT*, either because the *WSJ* and the *NYT* report similarly, or because the content of the *WP* is somehow complementary to the *WSJ*. Could there be a way to establish that one newspaper is more valuable than another, no matter what existing subscriptions the reader might have?

In this paper, we study the general version of this question. We derive comparisons of information sources that are robust to the presence of other information.

Formally, Blackwell models an information source as an *experiment*: a collection of possible outcomes and a conditional distribution of outcomes given the state. Blackwell's foundational result is that experiment A is more valuable than experiment B, regardless of the decision problem (i.e., the agent's action set and preferences over actions and states), if and only if the distribution of beliefs induced by observing A is a mean-preserving spread of that induced by observing B. Importantly, however, an experiment does not specify how observations from one information source are correlated with observations from other sources. This suffices for Blackwell's purpose, since he implicitly assumes that the two experiments being compared are the only information sources available to the agent.[2]

[1] Throughout this paper we focus on the instrumental value of information sources, ignoring the possibility that reading a newspaper might also provide pure consumption value (Ely *et al.*, 2015).

[2] Blackwell's result extends to situations where there might be other sources of information but only if those sources

In order to capture the joint informational content of multiple sources, we follow Green and Stokey (1978) in modeling an information source as a *signal*: a partition of an expanded state space $\Omega \times X$ that distinguishes payoff-relevant states (Ω) from those that govern the realization of observations conditional on the state (X). A signal induces an experiment, but it also pins down its correlation with other signals. In particular, the information generated by observing signals A and B is given by the join of the partitions, denoted $A \vee B$.

We say that signal A *Blackwell dominates* signal B if the experiment induced by A is more valuable than an experiment induced by B, regardless of the decision problem.[3] We then introduce the strong Blackwell order, defined as follows: signal A *strongly Blackwell dominates* signal B if for every signal C, $A \vee C$ Blackwell dominates $B \vee C$. In other words, we extend Blackwell's agnosticism about the agent's preferences to agnosticism about what other information the agent might have.

Our first theorem characterizes the strong Blackwell order. Say that signal A *reveals-or-refines* signal B if every signal realization of A either (i) occurs in only one state (and thus "reveals" the state), or (ii) is a subset of some signal realization of B (and thus "refines" B, pinning down what information is observed by B).[4] We show that A strongly Blackwell dominates B if and only if A reveals-or-refines B.

Having established this result, we turn to comparisons of information sources distinct from Blackwell. These additional comparisons can be put into two broad categories.

The first category includes comparisons that depend on the correlation across information sources, and is thus natural given our shift in focus from experiments to signals. Say that signal A is *sufficient* for signal B if B does not contain additional information about the state beyond that contained in A. Thus, for any agent with access to A, the marginal value of B is zero. For another comparison, say that signal A *martingale dominates* signal B if an agent who forms some posterior after observing B thinks that an agent who observes A will, in expectation, also hold that posterior.[5]

are conditionally independent of the two experiments being compared. This reflects the fact that the Blackwell comparison of experiments does not depend on prior beliefs.

[3] For most of our analysis, we treat the prior belief as fixed. As is well known, if one source of information Blackwell dominates another for some (interior) prior, then it does so for all priors. We discuss this at greater length in Section 4.2.

[4] So, if Alice observes A and Bob observes B, either Alice's first-order beliefs (about the state) or second-order beliefs (about Bob's beliefs) are degenerate.

[5] In Section 4.2, we discuss the sense in which martingale dominance captures a notion of being "more informative."

The second category consists of comparisons that weaken the Blackwell order and only require that one information source be more valuable than another on some subset of decision problems. An important example of such restricted Blackwell comparisons is the Lehmann (1988) order, which is derived from the set of monotone decision problems.

Like Blackwell, each of the aforementioned comparisons (sufficiency, martingale, Lehmann) implicitly presumes that the agent has no additional sources of information. Analogously to the notion of strong Blackwell dominance, however, it is possible to strengthen any comparisons of signals to reflect robustness to the presence of other information. Formally, given a relation $\mathcal{P}$ on signals, let the strengthening of $\mathcal{P}$, denoted $\overline{\mathcal{P}}$, be defined as $A\overline{\mathcal{P}}B$ if for any C, $(A \vee C)\,\mathcal{P}\,(B \vee C)$.

We first generalize our characterization of strong Blackwell dominance and establish that, as long as the subset of decision problems is sufficiently rich,[6] the strong version of the restricted Blackwell comparisons is also equal to reveal-or-refine. In particular, the strong version of the Lehmann order is equal to reveal-or-refine.

We also show that strong Blackwell implies sufficiency, which in turn implies martingale, which in turn implies Blackwell. This fact, combined with some basic properties of strengthening (namely monotonicity and idempotence), delivers the result that the strong versions of both sufficiency and martingale are also equal to reveal-or-refine.

Broadly speaking, there are various relations on information sources. Our analysis highlights sufficiency, martingale, Blackwell, and Lehmann. Even though these are all distinct, their strong versions – which make the comparisons robust to additional information – coincide. The strong versions of all of them are equal to reveal-or-refine.

Our paper is most closely related to the literature on ordinal comparisons of the ex-ante value of information sources, starting with Blackwell (1951).[7] Much of this research focuses on ways to weaken the Blackwell order. Lehmann (1988), Persico (2000), and Athey and Levin (2018) consider comparisons that apply to a subset of decision problems and/or a subset of experiments. Moscarini and Smith (2002) and Mu *et al.* (2021b) compare the values of large numbers of independent draws

For some intuition, suppose that A reveals the state but B does not. It is then easy to see that A martingale dominates B, but B does not martingale dominate A.

[6] In particular, we require that for any two states, the subset includes a decision problem under which it is valuable to distinguish those two states.

[7] A smaller literature considers the ex-post value of information (Frankel and Kamenica (2019); Frankel and Kasy (2022)).

of different experiments.

Another closely related literature studies the joint informational content of multiple information sources.[8] Börgers *et al.* (2013) consider the question of when signals are complements or substitutes. Gentzkow and Kamenica (2017a,b) consider the impact of competition when multiple senders provide potentially correlated signals in an attempt to influence a receiver. Liang and Mu (2020) and Liang *et al.* (2022) consider acquisition of potentially complementary information sources. Brooks *et al.* (2022) analyze the relationship between the comparison of information sources conceptualized as experiments, according to the Blackwell order, and the comparison of information sources conceptualized as signals, according to refinement, sufficiency, and martingale.[9] Specifically, they ask when a collection of Blackwell-ordered experiments can be induced by a collection of refinement-, sufficiency-, or martingale-ordered signals.

2 Signals and experiments

There is a finite state space Ω and an interior prior $\mu_0 \in \Delta\Omega$. We denote a typical state by ω.

An *experiment* τ is a distribution of beliefs – i.e., an element of $\Delta\Delta\Omega$ – that has finite support and satisfies $\mathbb{E}_\tau[\mu] = \mu_0$. (An alternative definition of an experiment is a map from Ω to distributions over signal realizations, but as is common, we simply identify each experiment with the distribution of beliefs it induces.) We write $\tau \succsim \tau'$ if τ is a mean-preserving spread of τ'.

A *signal* π is a finite partition of $\Omega \times [0,1]$ s.t. $\pi \subset S$, where S is the set of non-empty Lebesgue-measurable subsets of $\Omega \times [0,1]$ (Green and Stokey, 1978; Gentzkow and Kamenica, 2017a).[10] An element $s \in S$ is a *signal realization*. The interpretation of this formalism is that a random variable x, drawn uniformly from $[0,1]$, determines the signal realization conditional on the state. Thus, the conditional probability of s given ω is $p^\omega(s) = \lambda(\{x | (\omega, x) \in s\})$ where $\lambda(\cdot)$ denotes the Lebesgue

[8] Just as additional sources of information can alter the value of a signal, additional sources of income can alter the value of a monetary gamble. Mu *et al.* (2021a) explore how a decision maker's preferences over monetary gambles can depend on background risk, i.e., independent uncertainty over income.

[9] As we discuss in Section 4.2, the notion of martingale relation introduced in Brooks *et al.* (2022) is slightly different from the one we study here.

[10] Green and Stokey (1978; 2022) introduce the notion of signals as partitions of an expanded state space. The particular formalism we use here, and the accompanying graphical representation, was introduced by Gentzkow and Kamenica (2017a). In Section 6, we discuss how our results can be presented using the language of type spaces instead.

measure. Observing signal realization s induces the posterior μ_s.[11]

Given signal π, let $\tilde{s}_\pi$ be the associated S-valued random variable on $\Omega \times [0,1]$ induced by π.[12] Let $\tilde{\mu}_\pi \equiv \mu_{\tilde{s}_\pi}$ denote the associated belief-valued random variable that reflects the posterior induced by observing the realization from π. Finally, let $\langle \pi \rangle$ denote the distribution of $\tilde{\mu}_\pi$, i.e., the experiment induced by signal π. If $\langle \pi \rangle = \langle \pi' \rangle$, we say that π and π' are *Blackwell equivalent* and write $\pi \sim \pi'$.

We denote the set of all signals by Π. We say π *refines* π' and write $\pi \mathcal{R} \pi'$ if every element of π is a subset of some element of π'.[13] If $\pi \mathcal{R} \pi'$, an agent who observes π has access to all the information available to an agent who observes π'. The relation $\mathcal{R}$ is a partial order on Π and poset $(\Pi, \mathcal{R})$ is a lattice. We let $\vee$ denote the *join*, i.e., $\pi \vee \pi'$ is the coarsest refinement of both π and π'. Observing both π and π' results in the signal $\pi \vee \pi'$.

Given two (binary) relations on signals, $\mathcal{P}$ and $\mathcal{P}'$, we denote that $\mathcal{P}$ implies $\mathcal{P}'$ (i.e., $\pi \mathcal{P} \pi' \Rightarrow \pi \mathcal{P}' \pi'$) by $\mathcal{P} \subseteq \mathcal{P}'$.[14] If $\mathcal{P}$ implies $\mathcal{P}'$ but not vice versa, we have $\mathcal{P} \subsetneq \mathcal{P}'$.

3 Strong Blackwell

3.1 Absence of other information

A *decision problem* $D = (A, u)$ consists of a compact action set A and a continuous utility function $u : A \times \Omega \to \mathbb{R}$. The *value* of an experiment τ in problem D is given by $\mathbb{E}_{\tilde{\mu} \sim \tau} \left[\max_{a \in A} \mathbb{E}_{\omega \sim \tilde{\mu}} u(a, \omega) \right]$; the value of signal π in problem D is the value of the induced experiment $\langle \pi \rangle$.[15] Blackwell's Theorem (1953) establishes that τ is more valuable than τ' for every decision problem if and only if $\tau \succsim \tau'$.

We are primarily interested in studying comparisons of signals, rather than experiments. We say that signal π *Blackwell dominates* signal π' and write $\pi \mathcal{B} \pi'$ if π has a weakly higher value than π' for every D. Hence, $\pi \mathcal{B} \pi'$ if and only if $\langle \pi \rangle \succsim \langle \pi' \rangle$.

[11] The posterior probability of ω given s is $\mu_s^\omega \equiv \frac{p^\omega(s)\mu_0^\omega}{\sum_{\omega' \in \Omega} p^{\omega'}(s)\mu_0^{\omega'}}$ as long as the unconditional probability of s, $\sum_{\omega' \in \Omega} p^{\omega'}(s)\mu_0^{\omega'}$, is strictly positive. The specification of μ_s when s has zero probability is irrelevant for our results.

[12] Recall that an S-valued random variable on $\Omega \times [0,1]$ is simply a function from $\Omega \times [0,1]$ to S; $\tilde{s}_\pi$ maps each (ω, x) to the signal realization $s \in S$ that contains (ω, x) in partition π.

[13] We then also say π' *coarsens* π.

[14] Recall that a binary relation on Π is a subset of $\Pi \times \Pi$, with $\pi \mathcal{P} \pi'$ denoting that $(\pi, \pi') \in \mathcal{P} \subseteq \Pi \times \Pi$.

[15] One can of course subtract the payoff under the prior from the definition of this value, but since that is constant it would not change any comparisons of experiments.

Note that $\mathcal{B}$ is a relation on signals, but it is not a partial order. While it is reflexive and transitive, and therefore a preorder, it is not antisymmetric: $\pi \mathcal{B} \pi'$ and $\pi' \mathcal{B} \pi$ implies that the two signals are Blackwell equivalent ($\pi \sim \pi'$) but does not imply that they are the same signal ($\pi = \pi'$).[16] As we will see down the line, some economically meaningful relations on signals will not be transitive, so will not even be preorders.

3.2 Robustness to additional information

In the previous subsection, the analyst who compares the value of two signals is completely agnostic about the preferences of the agent but is implicitly dogmatic in the view that the signals whose value is being considered will be the only information available to the agent. We now extend agnosticism about preferences to agnosticism about what other information the agent observes.

An *extended decision problem* $\hat{D} = (A, u, \hat{\pi})$ consists of a compact action set A, a continuous utility function $u : A \times \Omega \to \mathbb{R}$, and a signal $\hat{\pi}$. The interpretation is that we are considering an agent with action set A and utility function u who has observed signal $\hat{\pi}$.

The value of a signal π in extended problem $\hat{D}$ is given by $\mathbb{E}_{\tilde{\mu} \sim \langle \pi \vee \hat{\pi} \rangle} \left[\max_{a \in A} \mathbb{E}_{\omega \sim \tilde{\mu}} u(a, \omega) \right]$. We say that signal π *strongly Blackwell dominates* signal π', denoted $\pi \overline{\mathcal{B}} \pi'$, if π has a higher value than π' for every extended decision problem $\hat{D}$.[17] We now describe some key properties of the strong Blackwell relation.

Remark 3.1. Strong Blackwell dominance implies but is not equivalent to Blackwell dominance, i.e., $\overline{\mathcal{B}} \subsetneq \mathcal{B}$. The fact that $\overline{\mathcal{B}} \subseteq \mathcal{B}$ follows from the observation that every extended decision problem is also a decision problem – we simply set $\hat{\pi}$ to be the trivial partition $\underline{\pi}$.[18] To see that $\overline{\mathcal{B}} \neq \mathcal{B}$, consider the three signals in Figure 1. It is easy to see that $\pi \mathcal{B} \pi'$ since π is informative about the state and π' is not. But, it is not the case that $\pi \overline{\mathcal{B}} \pi'$ since $\pi \vee \hat{\pi}$ is only partially informative about the state while $\pi' \vee \hat{\pi}$ fully reveals the state.

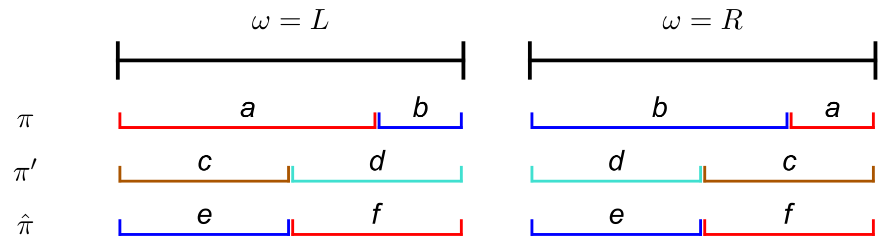
Remark 3.2. There are two other natural ways we could ask whether the comparison of two signals is influenced by the presence of additional information. First, we could ask whether π is neces-

[16] The Blackwell order on *experiments* – that is, the mean-preserving spread order – is of course a partial order.

[17] It is possible to formulate the Blackwell order on experiments without a reference to utility functions: experiment τ Blackwell dominates τ' if the set of feasible joint distributions of actions and states under τ is a superset of that under τ'. We could similarly formulate the notion of strong Blackwell dominance in terms of feasible joint distributions; our results would remain unchanged.

[18] The trivial partition is the one that contains a single signal realization, i.e., $\underline{\pi} = \{\Omega \times [0,1]\}$.

Figure 1: Blackwell vs. Strong Blackwell



It holds that $\pi\mathcal{B}\pi'$, but not that $\pi\overline{\mathcal{B}}\pi'$ because $\pi\vee\hat{\pi}$ does not Blackwell dominate $\pi'\vee\hat{\pi}$; in fact, $\pi'\vee\hat{\pi}$ fully reveals the state but $\pi\vee\hat{\pi}$ does not.

sarily more valuable than π' if the agent had observed a given signal *realization*. This would be an interim notion of more valuable, in contrast to the ex ante notion that is embodied in our definition of strong Blackwell. Formally, we could require that $\mathbb{E}_{\tilde{\mu}\sim\langle\pi|s\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right] \geq \mathbb{E}_{\tilde{\mu}\sim\langle\pi'|s\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right]$ for any triplet (A,u,s), where $\langle\pi|s\rangle$ denotes the distribution of posteriors induced by observing signal π after having previously observed signal realization s.

Second, we could consider the possibility that the agent, after obtaining a signal whose value we are interested in, could endogenously acquire additional costly information. Formally, we could require that $\sup_{\hat{\pi}\in\Pi}\mathbb{E}_{\tilde{\mu}\sim\langle\pi\vee\hat{\pi}\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right]-c\left(\hat{\pi}\right) \geq \sup_{\hat{\pi}\in\Pi}\mathbb{E}_{\tilde{\mu}\sim\langle\pi'\vee\hat{\pi}\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right]-c\left(\hat{\pi}\right)$ for any triplet (A,u,c), where $c:\Pi\to\mathbb{R}_+$ denotes the cost of acquiring additional information.

It turns out, however, that both of these alternative notions are equivalent to our definition of strong Blackwell dominance! We formalize and prove this claim in Appendix A.1.

Remark 3.3. Strong Blackwell dominance is transitive since Blackwell dominance is.

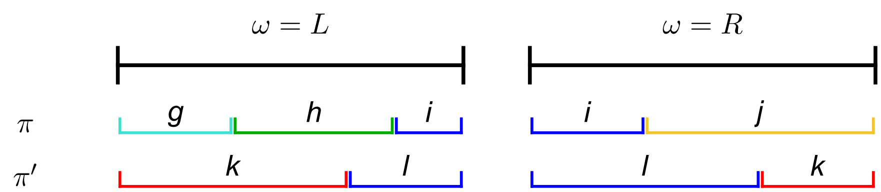
Our main result for this section is a characterization of strong Blackwell. This characterization can be motivated by considering two sufficient conditions for strong Blackwell.

First, it is immediate that refinement implies strong Blackwell: $\pi\mathcal{R}\pi'$ implies that $\langle\pi\vee\hat{\pi}\rangle \succsim \langle\pi'\vee\hat{\pi}\rangle$ for any $\hat{\pi}$. Second, consider any signal π that always reveals the state. It is immediate that $\pi\overline{\mathcal{B}}\pi'$ for every π': for any $\hat{\pi}$, the join $\pi\vee\hat{\pi}$ also always reveals the state, and therefore Blackwell dominates any other signal, including $\pi'\vee\hat{\pi}$.

Of course, neither of these sufficient conditions is necessary. Indeed, π' is strongly Blackwell dominated by any refinement of π', even if that refinement does not reveal the state, and π' is strongly Blackwell dominated by any signal that always reveals the state, even if that signal does not refine π'. Moreover, it might be that $\pi\overline{\mathcal{B}}\pi'$ even though π neither refines π' nor always reveals the state. The key insight is that if we consider elements of π one signal realization at a time, if it turns out that every signal realization of π either pins down the state or pins down the signal realization generated by π' (or both), then π must be more valuable than π' no matter what other information is available. Moreover, this condition is not merely sufficient for $\pi\overline{\mathcal{B}}\pi'$, it is also necessary.

Formally, say that π *reveals-or-refines* π', denoted $\pi\mathcal{O}\pi'$, if for every $s \in \pi$ either: (i) s reveals the state (i.e., $p(s|\omega) > 0$ for at most one ω), or (ii) $s \subseteq s'$ for some $s' \in \pi'$. See Figure 2 for an illustration. We then have the following characterization.

Figure 2: Reveal-or-refine



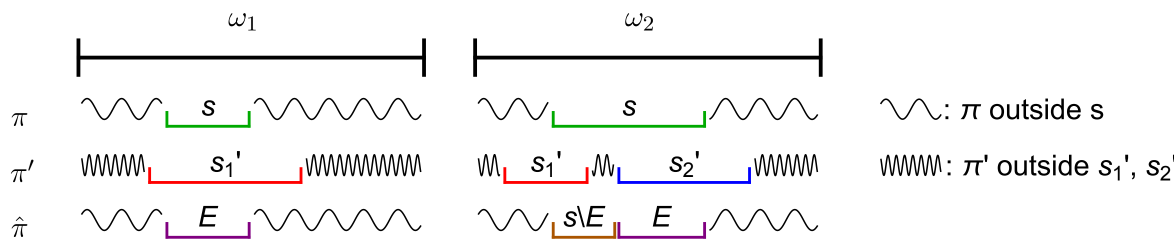
Signal π reveals-or-refines π', but it is not the case that π refines π' or that π always reveals the state.

Theorem 1. *Signal π strongly Blackwell dominates signal π' if and only if π reveals-or-refines π'. That is, $\overline{\mathcal{B}} = \mathcal{O}$.*

Proof. To see why reveal-or-refine implies strong Blackwell, first fix any extended decision problem. In order to show that π is more valuable than π', it suffices to show that π is more valuable than π' conditional on any signal realization s from π. If s reveals the state, nothing can be more valuable than π. If s refines $s' \in \pi'$, i.e., $s \subseteq s'$, then for any signal realization $\hat{s} \in \hat{\pi}$, $s \cap \hat{s} \subseteq s' \cap \hat{s}$, and thus π is more valuable than π'.

For the converse, suppose π does not reveal-or-refine π'. We can then find signal realizations s in π and $s'_1 \neq s'_2$ in π' along with states $\omega_1 \neq \omega_2$ such that $s \cap s'_1$ occurs with positive probability in ω_1 while $s \cap s'_2$ occurs with positive probability in ω_2. Let E be the event that either $\omega = \omega_1$ and $(\omega, x) \in s \cap s'_1$, or $\omega = \omega_2$ and $(\omega, x) \in s \cap s'_2$. Finally, let $\hat{\pi}$ be the join of π and a signal that reveals whether event E occurs. (See Figure 3 for an illustration.) Then $\pi \vee \hat{\pi} = \hat{\pi}$, while $\pi' \vee \hat{\pi}$ is strictly more informative than $\hat{\pi}$. In particular, in event E, signal $\hat{\pi}$ reveals only that event E has occurred, whereas $\pi' \vee \hat{\pi}$ additionally reveals whether the state is ω_1 (realization $s'_1 \cap E$) or ω_2 (realization $s'_2 \cap E$). $\square$

Figure 3: Construction of $\hat{\pi}$ in the proof of Theorem 1



Event E is the union of $s \cap s'_1$ in state ω_1 and $s \cap s'_2$ in state ω_2. Signal $\hat{\pi}$ is then defined as $\pi \vee \{E, E^c\}$.

The argument above establishes a result that is somewhat stronger than Theorem 1. It shows that if π does not reveal-or-refine π', then there is a $\hat{\pi}$ such that $\pi' \vee \hat{\pi}$ strictly Blackwell dominates $\pi \vee \hat{\pi}$. It is not merely that $\pi \vee \hat{\pi}$ is not comparable to $\pi' \vee \hat{\pi}$.

Moreover, as we discuss in Section 5, a similar argument can be used to provide a generalization of Theorem 1 that characterizes how to strengthen some relations on signals other than Blackwell.

A key qualitative insight from Theorem 1 is that even though the definition of strong Blackwell involves a universal quantification over all decision problems and all signals, the universal quantifier can in fact be eliminated, and strong Blackwell is reduced to the much simpler reveal-or-refine comparison, which only requires checking a condition for each of the (finitely many) signal realizations. Indeed, using a graphical representation of signals, it is straightforward to check whether one signal reveals-or-refines another via "visual inspection." For example, to compare π and π' in Figure 2, we consider each signal realization of π in turn. Realization $g \in \pi$ both reveals the state and refines realization $k \in \pi'$ (i.e., $g \subseteq k$); realization h reveals the state; realization i refines l; finally, j reveals the state. Thus, π reveals-or-refines π'.

4 Beyond Blackwell

In this section, we discuss other economically meaningful ways to compare signals. We first introduce two additional comparisons that require the shift in focus from experiments to signals since they concern the joint informational content of multiple sources of information. These two comparisons, sufficiency and martingale, turn out to be more demanding than Blackwell. Then, we consider comparisons that weaken the Blackwell order by requiring that a signal be more valuable than another only on some subset of decision problems. Examples include the Lehmann (1988) order and comparisons based on a single, specific decision problem. This section will set the stage for Section 5 where we will strengthen each of these comparisons to make them robust to the potential presence of other information, as we did previously for the Blackwell order.

4.1 Sufficiency

The Blackwell order is concerned with whether one source of information (e.g., NYT) is more valuable than another (e.g., *WP)*. Another meaningful question is when one source of information might make another source of information moot. For example, how could we tell whether a subscription to some newspaper is worthless given an agent's existing subscriptions?

Formally, we say that π is *sufficient* for π', denoted $\pi\mathcal{S}\pi'$, if in any decision problem the value of signal $\pi \vee \pi'$ is the same as value of signal π.[19]

This notion of sufficiency appears in various economic applications. For instance, Holmström (1979) shows that information about an agent's effort in a moral hazard problem is valuable if and only if the observable output is not sufficient for that information.

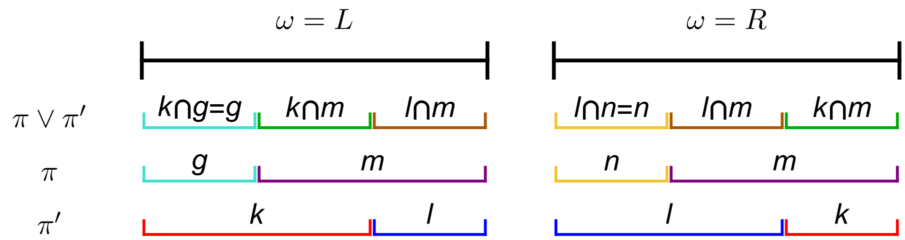
Remark 4.1. Signal π is sufficient for signal π' if and only if $(\pi \vee \pi') \sim \pi$. If $(\pi \vee \pi') \sim \pi$, then the value of $\pi \vee \pi'$ is the same as value of π for any decision problem, so $\pi\mathcal{S}\pi'$. Conversely, if $\pi\mathcal{S}\pi'$, then the fact that π alone yields as much value as $\pi \vee \pi'$ implies that $\pi\mathcal{B}(\pi \vee \pi')$. Since we know $(\pi \vee \pi')\mathcal{B}\pi$, we have that $(\pi \vee \pi') \sim \pi$.

Yet another equivalent formulation of sufficiency is in terms of the induced random variables: $\pi\mathcal{S}\pi' \Leftrightarrow \tilde{\mu}_{\pi\vee\pi'} = \tilde{\mu}_\pi$. In general, $\tilde{\mu}_\pi = \tilde{\mu}_{\pi'} \Rightarrow \pi \sim \pi'$ but $\pi \sim \pi' \not\Rightarrow \tilde{\mu}_\pi = \tilde{\mu}_{\pi'}$. That said, we do have that $\pi\mathcal{S}\pi' \Leftrightarrow \pi \vee \pi' \sim \pi \Leftrightarrow \tilde{\mu}_{\pi\vee\pi'} = \tilde{\mu}_\pi$. This equivalence follows from the more general result that if π^* refines π and $\pi^* \sim \pi$, then $\tilde{\mu}_{\pi^*} = \tilde{\mu}_\pi$.

The formulation of sufficiency in terms of random variables provides a simple way to check whether one signal is sufficient for another. To compare π and π' in Figure 4, we consider each signal realization of $\pi \vee \pi'$ in turn. Realization $g = k \cap g \in \pi \vee \pi'$ clearly leads to the same belief

[19] Alonso and Câmara (2018) and Brooks *et al.* (2022) use the terminology that π' is (statistically) redundant given π when π is sufficient for π'.

Figure 4: Checking for sufficiency



One can confirm that $\pi \mathcal{S} \pi'$ by comparing the likelihood ratios of each signal realization in π to the likelihood ratios of the overlapping signal realizations in $\pi \vee \pi'$.

as $g \in \pi$; realization $k \cap m \in \pi \vee \pi'$ leads to the same belief as $m \in \pi$ since

$$\frac{Pr\left(k \cap m | \omega = L\right)}{Pr\left(k \cap m | \omega = R\right)} = \frac{Pr\left(m | \omega = L\right)}{Pr\left(m | \omega = R\right)}.$$

The same is true for $l \cap m$ and m, and hence $\pi \mathcal{S} \pi'$.

Remark 4.2. Another equivalent definition of sufficiency is: for all $s \in \pi$ and all $s' \in \pi'$, $Pr\left(s'|s,\omega\right)$ is independent of ω. This formulation echoes Blackwell's (1953) notion of a garbling.[20] But unlike Blackwell, we have specified the underlying probability space, so we are not asking whether there exists a garbling that transforms experiment $\langle \pi \rangle$ into $\langle \pi' \rangle$. Rather, we ask whether – given their underlying correlation – the signal π' adds information about the state given signal π. Relatedly, it is worth noting that the following three conditions are equivalent: (i) $\pi \mathcal{B} \pi'$, (ii) $\exists \pi^*$ s.t. $\pi \sim \pi^*$ and $\pi^* \mathcal{R} \pi'$, and (iii) $\exists \pi^*$ s.t. $\pi \sim \pi^*$ and $\pi^* \mathcal{S} \pi'$. The equivalence of (i) and (ii) is Theorem 1 in Green and Stokey (1978).[21] The equivalence of (i) and (iii) is closely related to a standard formulation of Blackwell's theorem.

Remark 4.3. $\overline{\mathcal{B}} \subsetneq \mathcal{S} \subsetneq \mathcal{B}$.

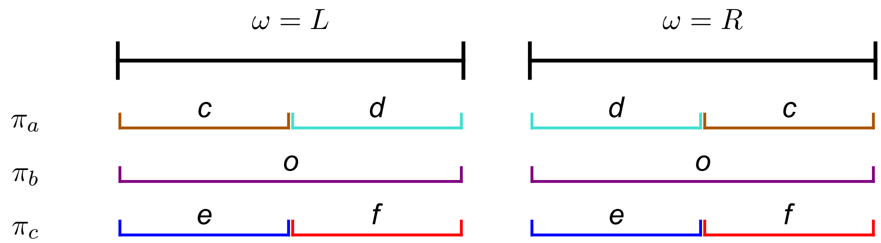
First, it is easy to see that $\overline{\mathcal{B}} \subseteq \mathcal{S}$. If $\pi \overline{\mathcal{B}} \pi'$, we know $\left(\pi \vee \hat{\pi}\right) \mathcal{B} \left(\pi' \vee \hat{\pi}\right)$ for any $\hat{\pi}$, including $\hat{\pi} = \pi$; hence, $\pi \mathcal{B} \left(\pi \vee \pi'\right)$. Since it's always the case that $\left(\pi \vee \pi'\right) \mathcal{B} \pi$, we have $\pi \sim \left(\pi \vee \pi'\right)$. Hence, $\pi \overline{\mathcal{B}} \pi'$ implies $\pi \mathcal{S} \pi'$.

Importantly, however, $\overline{\mathcal{B}} \neq \mathcal{S}$. This can be seen in Figure 6 where $\pi_{\mathcal{S}} \mathcal{S} \pi_0$ but $\neg \left(\pi_{\mathcal{S}} \overline{\mathcal{B}} \pi_0\right)$. The fact that $\overline{\mathcal{B}} \neq \mathcal{S}$ has a substantive economic interpretation. Suppose we know that $\pi \mathcal{B} \pi'$ but are

[20]This formulation also clarifies the relationship between our definition of sufficiency and the notion of a sufficient statistic in the field of statistics. Given some data $\vec{x}$, recall that a function $t\left(\vec{x}\right)$ is a sufficient statistic for ω if $Pr\left(\vec{x}|t\left(\vec{x}\right),\omega\right)$ is independent of ω.

[21]It is stated and proved using the formalism in our paper by Gentzkow and Kamenica (2017a).

Figure 5: Sufficiency is not transitive



It holds that $\pi_a \mathcal{S} \pi_b$ and $\pi_b \mathcal{S} \pi_c$ but not that $\pi_a \mathcal{S} \pi_c$.

not sure whether π remains more valuable than π' in the presence of some additional information $\hat{\pi}$. One might think that the "worst case" scenario would be if $\hat{\pi} = \pi$ (e.g., in comparing the value of NYT to the value of WP, we worry that the reader already has a subscription to the NYT). This scenario, however, only tells us, however, whether $\pi \mathcal{S} \pi'$, which is a weaker condition than $\pi \overline{\mathcal{B}} \pi'$. Thus, $\hat{\pi} = \pi$ is not the most stringent test-case for strong Blackwell dominance. Instead, a more consequential case is when $\hat{\pi}$ is complementary to π'.

It is also easy to see that $\mathcal{S} \subseteq \mathcal{B}$ since $\pi \mathcal{S} \pi'$ means the value of π in any decision problem is the same as the value of $\pi \vee \pi'$, which in turn must be weakly higher than the value of π'. Moreover, Figure 6 establishes that $\mathcal{S} \neq \mathcal{B}$ since $\pi_{\mathcal{B}} \mathcal{B} \pi_0$ but $\neg (\pi_{\mathcal{B}} \mathcal{S} \pi_0)$.

Remark 4.4. Sufficiency is not transitive. Consider Figure 5. Since $\pi_a \vee \pi_b = \pi_a$, π_a is *a fortiori* sufficient for π_b. Since both π_b and $\pi_b \vee \pi_c$ provide no information about the state, we have that π_b is sufficient for π_c. Yet, π_a is *not* sufficient for π_c; π_a on its own provides no information about the state while $\pi_a \vee \pi_c$ fully reveals the state.

4.2 Martingale

A widely used and basic observation in information economics is that "beliefs are a martingale." If an agent with some current belief μ_0 observes additional data from some source of information, their expected posterior belief must be μ_0. This is a consequence of the Law of Iterated Expectations.

In the context of signals, one way to formulate this observation is to note that if π refines π', then it must be the case that $\mathbb{E}\left[\tilde{\mu}_\pi | \tilde{s}_{\pi'}\right] = \tilde{\mu}_{\pi'}$. In other words, additional information cannot change beliefs on average.

In this paper, we take a novel perspective on the martingale property. Instead of treating it as an implication of Bayesian updating, we consider it as a relation between sources of information. When is it the case that, if I read the *Washington Post*, I think that in expectation, a reader of the *New York Times* would hold the same belief that I do? If WP were much more informative than NYT, there would be no reason to think this: a reader of WP might know the state of the world and yet expect the reader of NYT to remain uninformed. By contrast, if NYT contains all the information that WP does, then the WP reader would in fact think that the expected belief of the NYT reader is equal to their own. Thus, the martingale property ($\mathbb{E}\left[\tilde{\mu}_{\pi}|\tilde{s}_{\pi'}\right] = \tilde{\mu}_{\pi'}$) tells us that π is in some sense "more informative" than π'. In this section, we unpack what that means.

To formally define the martingale relation, we need to address a subtlety that was absent from the considerations of the Blackwell and sufficiency relations. We fixed an interior prior μ_0 at the outset, but as is well known, the Blackwell comparison is prior independent, so by extension the sufficiency comparison is as well. In other words, whether $\pi\mathcal{B}\pi'$ or whether $\pi\mathcal{S}\pi'$ does not depend on μ_0.

By contrast, for a given π and π', whether $\mathbb{E}\left[\tilde{\mu}_{\pi}|\tilde{s}_{\pi'}\right] = \tilde{\mu}_{\pi'}$ can depend on μ_0. (An example of this is given in Appendix A.3.)

Accordingly, we say π *martingale dominates* π', denoted $\pi\mathcal{M}\pi'$, if $\mathbb{E}\left[\tilde{\mu}_{\pi}|\tilde{s}_{\pi'}\right] = \tilde{\mu}_{\pi'}$ holds for any choice of μ_0.

Remark 4.5. In prior work (Brooks *et al.*, 2022), we introduced a similar relation, termed *belief-martingale,* defined by $\mathbb{E}\left[\tilde{\mu}_{\pi}|\tilde{\mu}_{\pi'}\right] = \tilde{\mu}_{\pi'}$. To understand the distinction between the two relations, it is helpful to introduce the idea of the belief-coarsening of a signal. Given any signal π, we let the *belief-coarsening* of π, denoted $C(\pi)$, be the signal that "pools together" any signal realizations in π that induce the same belief. Formally, $C(\pi)$ is the finest coarsening of π such that for any $s, s' \in C(\pi)$, $s \neq s' \Rightarrow \mu_s \neq \mu_{s'}$. With this definition in hand, we have that π belief-martingale dominates π' if and only if π martingale dominates $C(\pi')$.[22] Moreover, $\pi\mathcal{M}\pi'$ implies that π belief-martingale dominates π'.

Remark 4.6. The notion of belief-coarsening also provides another way to characterize the martin-

[22]For another illustration of belief-coarsening, if π refines $C(\pi')$, that means that an agent who observes π knows the first-order beliefs of an agent who observes π'. In Brooks *et al.* (2022), we discuss such "knowledge of first-order beliefs" as an example of a proper relation on signals, i.e., a relation that is implied by refinement and implies belief-martingale.

gale relation. In particular, it turns out that $\pi\mathcal{M}\pi'$ if and only if $C(\pi)\mathcal{S}\pi'$. It is easy to see that $C(\pi)\mathcal{S}\pi'$ implies $\pi\mathcal{M}\pi'$ because $C(\pi)\mathcal{S}\pi'$ implies $C(\pi)\mathcal{M}\pi'$ (because $\mathcal{S}\subseteq\mathcal{M}$) and $C(\pi)\mathcal{M}\pi'$ implies $\pi\mathcal{M}\pi'$ (because $\tilde{\mu}_{C(\pi)}=\tilde{\mu}_{\pi}$). The other direction is more subtle, and we provide a detailed argument in Appendix A.2. The equivalence of $\pi\mathcal{M}\pi'$ with $C(\pi)\mathcal{S}\pi'$ is illustrated in Figure 6 in Section 5, taking $\pi=\pi_{\mathcal{M}}$ and $\pi'=\pi_0$: $\pi_{\mathcal{M}}$ martingale dominates π_0 and $\pi_{\mathcal{M}}$ is not sufficient for π_0, but $C(\pi_{\mathcal{M}})$ is sufficient for π_0. More generally, the fact that $\pi\mathcal{M}\pi'\Leftrightarrow C(\pi)\mathcal{S}\pi'$ provides a simple way to check whether one signal martingale dominates another.

Remark 4.7. $\mathcal{S}\subsetneq\mathcal{M}\subsetneq\mathcal{B}$. To see that $\mathcal{S}\subseteq\mathcal{M}$, suppose $\pi\mathcal{S}\pi'$. Since $\mathcal{R}\subseteq\mathcal{M}$, we know that $(\pi\vee\pi')\mathcal{M}\pi'$, i.e., $\mathbb{E}[\tilde{\mu}_{\pi\vee\pi'}|\tilde{s}_{\pi'}]=\tilde{\mu}_{\pi'}$, which in turn implies $\mathbb{E}[\tilde{\mu}_{\pi}|\tilde{s}_{\pi'}]=\tilde{\mu}_{\pi'}$ since $\tilde{\mu}_{\pi\vee\pi'}=\tilde{\mu}_{\pi}$. Thus $\pi\mathcal{M}\pi'$. To see that $\mathcal{M}\subseteq\mathcal{B}$, note that $\mathbb{E}[\tilde{\mu}_{\pi}|\tilde{s}_{\pi'}]=\tilde{\mu}_{\pi'}$ implies that the distribution of $\tilde{\mu}_{\pi}$ is a mean-preserving spread of the distribution of $\tilde{\mu}_{\pi'}$. Another way to see that $\mathcal{S}\subseteq\mathcal{M}\subseteq\mathcal{B}$ is to note the following analogous characterizations of these three relations (as shown in Appendix A.4).

- $\pi\mathcal{S}\pi'$ if and only if $\exists\pi^*$ s.t. $C(\pi^*)=C(\pi)$, $\pi^*\mathcal{R}\pi$, and $\pi^*\mathcal{R}\pi'$. (We can take $\pi^*=\pi\vee\pi'$.)
- $\pi\mathcal{M}\pi'$ if and only if $\exists\pi^*$ s.t. $C(\pi^*)=C(\pi)$ and $\pi^*\mathcal{R}\pi'$.
- $\pi\mathcal{B}\pi'$ if and only if $\exists\pi^*$ s.t. $\pi^*\sim\pi$ and $\pi^*\mathcal{R}\pi'$ (as noted in Remark 4.2).[23]

To see that $\mathcal{S}\neq\mathcal{M}\neq\mathcal{B}$, see Figure 6 in Section 5. In the figure, we see that $\pi_{\mathcal{M}}\mathcal{M}\pi_0$ but $\neg(\pi_{\mathcal{M}}\mathcal{S}\pi_0)$, and that $\pi_{\mathcal{B}}\mathcal{B}\pi_0$ but $\neg(\pi_{\mathcal{B}}\mathcal{M}\pi_0)$.

Remark 4.8. The martingale relation is not transitive. See example in Figure 8 in Appendix A.5.

4.3 Restricted Blackwell

All of the comparisons we have introduced thus far (Strong Blackwell, sufficiency, martingale) are more restrictive than the Blackwell comparison. Yet, many have argued that the Blackwell comparison itself is already excessively restrictive (Lehmann, 1988; Moscarini and Smith, 2002). One approach to weakening the Blackwell comparison has been to consider a restricted class of decision problems (e.g., supermodular payoffs), often for a restricted class of experiments (e.g., those that satisfy the monotone likelihood ratio property). This approach was pioneered by Lehmann

[23]Since $C(\pi^*)=C(\pi)$ implies that $\pi^*\sim\pi$, these characterizations make it clear that $\mathcal{M}\subseteq\mathcal{B}$.

(1988) and further developed by Persico (2000), Quah and Strulovici (2009), Cabrales *et al.* (2013), and Athey and Levin (2018). In this section, we consider comparisons of signals over restricted domains of decision problems.

Consider some class of decision problems $\mathbb{D}$. We say that signal π is *more valuable for* $\mathbb{D}$ than π', denoted $\pi \mathcal{V}_{\mathbb{D}} \pi'$, if π has a weakly higher value than π' for every $D \in \mathbb{D}$. We refer to these as restricted Blackwell relations. If $\mathbb{D}$ is the class of all decision problems, then $\mathcal{V}_{\mathbb{D}}$ is the Blackwell relation $\mathcal{B}$.

Suppose that the set of states Ω is ordered and $\mathbb{D}$ consists of decision problems (A, u) that are monotone in the sense that: (i) A is ordered, (ii) $a^* (\omega) \equiv \arg\max_{a \in A} u(a, \omega)$ is single-valued and non-decreasing in ω, and (iii) $\overline{\overline{a}} \geq \overline{a} \geq a^*(\omega) \geq \underline{a} \geq \underline{\underline{a}}$ implies $u\left(\overline{\overline{a}}, \omega\right) \leq u(\overline{a}, \omega)$ and $u\left(\underline{\underline{a}}, \omega\right) \leq u(\underline{a}, \omega)$. Lehmann (1988) studies the value of experiments for problems in this domain. Accordingly, we denote $\mathcal{V}_{\mathbb{D}}$ for this domain by $\mathcal{L}$.[24]

We could also consider a case where $\mathbb{D}$ is a singleton, containing only some specific decision problem D. In that case, $\mathcal{V}_{\mathbb{D}}$ is complete: for any two signals π and π', we have $\pi \mathcal{V}_{\mathbb{D}} \pi'$ or $\pi' \mathcal{V}_{\mathbb{D}} \pi$.[25]

While $\mathcal{V}_{\mathbb{D}}$ is well-defined for any domain $\mathbb{D}$, we will be particularly interested in domains that are sufficiently rich, in the following sense. We say that $\mathbb{D}$ is *discriminating* if for any distinct $\omega, \omega' \in \Omega$, there exists $(A, u) \in \mathbb{D}$ such that $\arg\max_{a \in A} u(a, \omega)$ and $\arg\max_{a \in A} u(a, \omega')$ do not intersect. In other words, for any pair of states, there is a decision problem in $\mathbb{D}$ that benefits from distinguishing those two states. In particular, the domains that induce the Blackwell and the Lehmann relations are discriminating.

Our goal in this paper is not to characterize $\mathcal{V}_{\mathbb{D}}$ for various domains $\mathbb{D}$. Instead, we will present a general result about how to strengthen $\mathcal{V}_{\mathbb{D}}$ (for any discriminating $\mathbb{D}$) to make it robust to the potential presence of other information.

[24] Lehmann's (1988) main result considers two signals π_a and π_b that satisfy the monotone likelihood ratio in s for some total order on S. For this case, Lehmann proves that $\pi_a \mathcal{L} \pi_b$ if and only if $\left(F_\omega^{\pi_a}\right)^{-1} \left(F_\omega^{\pi_b}(s)\right)$ is a non-decreasing function of ω for each s, where $F_\omega^{\pi}(s)$ is the probability that π yields a realization lower than s in ω. Many authors take this quantile condition to be the *definition* of the Lehmann order. In contrast, we define $\mathcal{L}$ solely in terms of the restriction on the set of (A, u) pairs being considered.

[25] Recall that we are considering a fixed prior on Ω; hence the value of each signal for D is simply a number.

5 Generalized strengthening

We now formalize how to strengthen an arbitrary relation to make it robust to the presence of additional information. Given a relation $\mathcal{P}$ on Π, let *strong* $\mathcal{P}$, denoted $\overline{\mathcal{P}}$, be defined by $\pi\overline{\mathcal{P}}\pi'$ if $(\pi \vee \hat{\pi})\,\mathcal{P}\,(\pi' \vee \hat{\pi})$ for all $\hat{\pi} \in \Pi$.

We have already seen one important case of the strong version of a relation, the strong Blackwell order $\overline{\mathcal{B}}$. In fact, the proof of Theorem 1 can easily be extended to establish a more general result:

Theorem 1′. *Suppose that $\mathbb{D}$ is discriminating. Then, $\pi\overline{\mathcal{V}}_{\mathbb{D}}\pi'$ if and only if π reveals-or-refines π'. In particular, $\overline{\mathcal{L}} = \mathcal{O}$.*

Proof. Theorem 1 establishes that when π reveals-or-refines π', π is more valuable than π' for any extended decision problem $(A, u, \hat{\pi})$. *A fortiori*, π is more valuable than π' for any $(A, u, \hat{\pi})$ such that $(A, u) \in \mathbb{D}$. Therefore, π reveals-or-refines π' implies $\pi\overline{\mathcal{V}}_{\mathbb{D}}\pi'$ for any $\mathbb{D}$.

Now suppose that $\mathbb{D}$ is discriminating. We wish to show $\pi\overline{\mathcal{V}}_{\mathbb{D}}\pi'$ implies π reveals-or-refines π'. Consider signals π and π' such that π does not reveal-or-refine π'. In the proof of Theorem 1, we observed that this implies that there are distinct states ω_1 and ω_2, an event E, and a signal $\hat{\pi}$ such that: (i) $\pi \vee \hat{\pi} = \hat{\pi}$ and (ii) $\pi' \vee \hat{\pi}$ reveals $\hat{\pi}$ and, in event E, also reveals whether the state is ω_1 or ω_2. Because $\mathbb{D}$ is discriminating, there is a decision problem $(A, u) \in \mathbb{D}$ such that it is strictly valuable to distinguish ω_1 and ω_2, implying that $\pi \vee \hat{\pi}$ is less valuable than $\pi' \vee \hat{\pi}$ for a decision problem in $\mathbb{D}$. Therefore, it is not the case that $\pi\overline{\mathcal{V}}_{\mathbb{D}}\pi'$. $\square$

Remark 5.1. The assumption that $\mathbb{D}$ is discriminating is necessary for Theorem **1′**. In Appendix A.6, we show that for any non-discriminating $\mathbb{D}$, we have $\overline{\mathcal{V}}_{\mathbb{D}} \neq \mathcal{O}$. Thus, $\mathbb{D}$ is discriminating if and only if $\overline{\mathcal{V}}_{\mathbb{D}} = \mathcal{O}$.

We now establish some basic facts about strengthening that apply no matter what relation is being strengthened.

(i) Strengthening strengthens: for any $\mathcal{P}$, we have $\overline{\mathcal{P}} \subseteq \mathcal{P}$. Proof: If $(\pi \vee \hat{\pi})\,\mathcal{P}\,(\pi' \vee \hat{\pi})$ for all $\hat{\pi}$, then $\pi = (\pi \vee \underline{\pi})\,\mathcal{P}\,(\pi' \vee \underline{\pi}) = \pi'$.

(ii) Strengthening is idempotent: for any $\mathcal{P}$, $\overline{\overline{\mathcal{P}}} = \overline{\mathcal{P}}$. Proof: From (i), $\overline{\overline{\mathcal{P}}} \subseteq \overline{\mathcal{P}}$. To show $\overline{\mathcal{P}} \subseteq \overline{\overline{\mathcal{P}}}$, suppose $\pi\overline{\mathcal{P}}\pi'$, i.e., $(\pi \vee \hat{\pi})\,\mathcal{P}\,(\pi' \vee \hat{\pi})$ for all $\hat{\pi} \in \Pi$. Then, for any $\hat{\pi}, \tilde{\pi} \in \Pi$, we have

$(\pi \vee \hat{\pi} \vee \tilde{\pi}) \, \mathcal{P} \, (\pi' \vee \hat{\pi} \vee \tilde{\pi})$ since $\hat{\pi} \vee \tilde{\pi} \in \Pi$.

(iii) Strengthening is monotone: if $\mathcal{P} \subseteq \mathcal{P}'$, then $\overline{\mathcal{P}} \subseteq \overline{\mathcal{P}'}$. Proof: Suppose $\mathcal{P} \subseteq \mathcal{P}'$ and $\pi \overline{\mathcal{P}} \pi'$. For any $\hat{\pi}$, we have that $(\pi \vee \hat{\pi}) \, \mathcal{P} \, (\pi' \vee \hat{\pi})$, which in turn implies $(\pi \vee \hat{\pi}) \, \mathcal{P}' \, (\pi' \vee \hat{\pi})$. Since this holds for all $\hat{\pi}$, we have that $\pi \overline{\mathcal{P}'} \pi'$.

The latter two properties yield a simple Lemma that will turn out to be very useful.[26]

Lemma 1. *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are two relations on signals such that $\overline{\mathcal{Q}} \subseteq \mathcal{P} \subseteq \mathcal{Q}$. Then, $\overline{\mathcal{P}} = \overline{\mathcal{Q}}$.*

Proof. $\overline{\mathcal{Q}} \subseteq \mathcal{P}$ implies $\overline{\overline{\mathcal{Q}}} \subseteq \overline{\mathcal{P}}$ (by monotonicity), which in turn implies $\overline{\mathcal{Q}} \subseteq \overline{\mathcal{P}}$ (by idempotency). $\mathcal{P} \subseteq \mathcal{Q}$ implies $\overline{\mathcal{P}} \subseteq \overline{\mathcal{Q}}$ (by monotonicity). Hence, $\overline{\mathcal{P}} = \overline{\mathcal{Q}}$. $\square$

With this Lemma in hand, it turns out to be very easy to characterize the strong versions of a large class of relations on signals.

Theorem 2. *Suppose $\mathcal{P}$ is a relation on signals and $\mathcal{O} \subseteq \mathcal{P} \subseteq \mathcal{V}_{\mathbb{D}}$ for some discriminating $\mathbb{D}$. Then, $\pi \overline{\mathcal{P}} \pi'$ if and only if π reveals-or-refines π'. In particular, $\overline{\mathcal{S}} = \overline{\mathcal{M}} = \mathcal{O}$.*

Proof. This follows from Theorem **1′** and Lemma 1, coupled with the fact that, if $\mathbb{D}$ is discriminating, $\mathcal{O} = \overline{\mathcal{B}} \subseteq \mathcal{S} \subseteq \mathcal{M} \subseteq \mathcal{B} \subseteq \mathcal{V}_{\mathbb{D}}$. $\square$

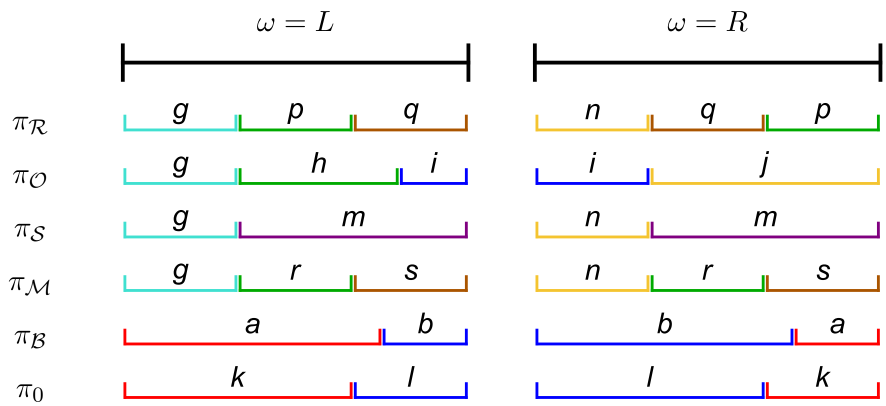
Putting together our results we have:

Corollary 1. $\overline{\mathcal{S}} = \overline{\mathcal{M}} = \overline{\mathcal{B}} = \overline{\mathcal{L}} = \mathcal{O}$.

There are various ways to compare the usefulness of a source of information – sufficiency, martingale, Blackwell, Lehmann. For any of these comparisons, we may wish to consider the strong version of the comparison that is robust to the potential presence of additional information. Our results deliver a remarkable message, namely that, even though sufficiency, martingale, Blackwell, and Lehmann are all distinct, their strong versions coincide! Moreover, the strong version of each of these comparisons is a simple relation, reveal-or-refine, which is very easy to check and involves no quantifiers over decision problems or signals.

[26] These two properties also appear in the notion of a closure operator in set theory. A function cl that maps sets to sets is a closure operator if for any two sets X and Y, we have: (i) $X \subseteq \mathrm{cl}(X)$, (ii) $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$, and (iii) $X \subseteq Y \Rightarrow \mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$. Property (i) is the "reverse" of property (i) in the remark above. Properties (ii) and (iii) are exactly idempotence and monotonicity. Consequently, an analogue of the Lemma below applies to any closure operator: if cl is a closure operator, then for any two sets X and Y, $X \subseteq Y \subseteq \mathrm{cl}(X)$ implies $\mathrm{cl}(X) = \mathrm{cl}(Y)$.

Figure 6: Ranking the relations



This figure illustrates the ranking of the relations $\mathcal{R} \subsetneq \mathcal{O} \subsetneq \mathcal{S} \subsetneq \mathcal{M} \subsetneq \mathcal{B}$, where for each relation $\mathcal{P}$, we have $\pi_{\mathcal{P}}\mathcal{P}\pi_0$. To confirm the strictness of this ranking, we see that $\pi_{\mathcal{O}}\mathcal{O}\pi_0$ but $\neg(\pi_{\mathcal{O}}\mathcal{R}\pi_0)$; $\pi_{\mathcal{S}}\mathcal{S}\pi_0$ but $\neg(\pi_{\mathcal{S}}\mathcal{O}\pi_0)$; $\pi_{\mathcal{M}}\mathcal{M}\pi_0$ but $\neg(\pi_{\mathcal{M}}\mathcal{S}\pi_0)$; and $\pi_{\mathcal{B}}\mathcal{B}\pi_0$ but $\neg(\pi_{\mathcal{B}}\mathcal{M}\pi_0)$. The fact that $\neg(\pi_{\mathcal{B}}\mathcal{M}\pi_0)$ follows from the fact that $C(\pi_{\mathcal{B}}) = \pi_{\mathcal{B}}$ is not sufficient for π_0.

One relation that we have discussed but is not covered by the theorems in this section is refinement. Refinement is more demanding than reveal-or-refine ($\mathcal{R} \subsetneq \mathcal{O}$). In fact, it is easy to see that refinement is unaffected by strengthening ($\overline{\mathcal{R}} = \mathcal{R}$): if π refines π' then $\pi \vee \hat{\pi}$ refines $\pi' \vee \hat{\pi}$ for any $\hat{\pi}$.

Summarizing the relations that we have considered, we can order them as follows:

$$\overline{\mathcal{R}} = \mathcal{R} \subsetneq \overline{\mathcal{S}} = \overline{\mathcal{M}} = \overline{\mathcal{B}} = \overline{\mathcal{L}} = \mathcal{O} \subsetneq \mathcal{S} \subsetneq \mathcal{M} \subsetneq \mathcal{B} \subsetneq \mathcal{L}.$$

Figure 6 illustrates the strict comparisons, providing examples where signals are ranked by reveal-or-refine but not refinement; sufficiency but not reveal-or-refine; martingale but not sufficiency; and Blackwell but not martingale.[27]

6 Reformulation in terms of type spaces

Signals, i.e., partitions of $\Omega \times [0,1]$, are not the only (nor the most common) way of encoding correlations between information sources. A *type space* $(\boldsymbol{T}, Q)$ consists of a finite product set $\boldsymbol{T} = T_1 \times \cdots \times T_n$, and a probability distribution $Q \in \Delta(\Omega \times \boldsymbol{T})$ for which the marginal of Q

[27] In general, $\mathcal{B} \subsetneq \mathcal{L}$, but in the case of binary states, $\mathcal{B} = \mathcal{L}$. Consequently, we do not include comparisons with $\mathcal{L}$ in the figure.

on Ω is μ_0. The observation of type $t_i \in T_i$ pins down an experiment. The full distribution Q, moreover, specifies the overall correlation of types and thus the joint informational content of observing multiple types. Consequently, we could have formulated both our questions and our answers in terms of type space.[28]

Specifically, consider some type space $(T_1 \times T_2, Q)$. We say that $(T_1 \times T_2 \times T_3, Q')$ *extends* $(T_1 \times T_2, Q)$ if the probability of any (ω, t_1, t_2) is the same under Q and Q'. We would then say that t_1 Blackwell dominates t_2 if observing t_1 is more valuable than observing t_2 in any decision problem. Moreover, t_1 strongly Blackwell dominates t_2 if for every extension $(T_1 \times T_2 \times T_3, Q')$, observing (t_1, t_3) is more valuable than observing (t_2, t_3) in any decision problem. The analogue of our Theorem 1 is that under $(T_1 \times T_2, Q)$, t_1 strongly Blackwell dominates t_2 if and only if for every $t_1 \in T_1$, either: (i) there is at most one ω such that the probability of (t_1, ω) under Q is strictly positive, or (ii) there is at most one t_2 such that the probability of (t_1, t_2) under Q is strictly positive. Similarly, analogues of the notions of sufficiency and martingale, and of Theorem **2** can be presented within the type space formalism.

We personally find that the formalism in terms of signals is notationally less cumbersome. Moreover, we consider it helpful to define the underlying probability space at the outset, so that each source of information (i.e., a signal) can be fully specified without any explicit reference to other sources in information. As the paragraph above makes clear, this is not possible within the type space formalism.

One might worry, however, that the signal formalism requires unrealistic levels of information about how some particular information source, such as the *NYT*, fits into the $\Omega \times [0,1]$ state space. This is not a concern if we interpret the unit interval not as some true, physical dimension of the state space, but rather as a modeling device. Under the modeling device interpretation, however, two issues arise. First, can every type space be represented through signals? Second, even if so, it may be that the appropriate encoding of one information source as a signal may depend on what other information sources will be considered. For example, suppose we have data on the experiment induced by reading the *NYT*. We choose some arbitrary encoding of the *NYT* as a signal, matching the conditional probability of each *NYT* realization in each state. We then learn we also have to

[28]We thank Dirk Bergemann and Stephen Morris for encouraging us to make this connection.

encode the *WSJ*, capturing the correlation of the *WSJ* with the state and with the *NYT*. Might it be that we now have to reconsider our encoding of the *NYT* in order to allow for consistency with the *WSJ*?[29] In Brooks *et al.* (2024), we show that these concerns are unwarranted. Any type space can be represented as a collection of signals and the encoding of type spaces into signals can be done myopically, one information source at a time.

7 Conclusion

Experiments have long been considered the natural formalism for modeling information sources. As we and others have argued, this formalism is incomplete, in that the definition of different experiments does not specify how they interact with one another. In contrast, we model information sources as *signals*, which provide a complete description of the joint distribution of data from one information source and all others.

With this shift in focus from experiments to signals, a number of natural questions emerge. In Brooks *et al.* (2022), we investigate the conditions under which a partial order on the information content of experiments can be made consistent with an analogous ordering on signals. In the present paper, we compare the value of signals with a focus on the robustness to potential presence of other information. We also argue for the study of relations on signals beyond the familiar Blackwell and refinement orders, including sufficiency and martingale. But many questions remain. What other relations on signals may be useful and/or meaningful in economic applications? What are the decision- or game-theoretic foundations for the different relations? (Sufficiency, for example, has a simple characterization that one signal not add value to another in any decision problem; refinement may be relevant in games, when a player cares not only about the underlying state but also about what other players know; we do not know of natural foundations for the martingale relation.) What are economically reasonable ways to model the cost of acquiring signals? We leave these issues for future work.

[29]We thank Ian Ball for raising this question.

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A Appendix

A.1 Alternative formulations of strong Blackwell dominance

An interim decision problem $\hat{D}^i = (A, u, s)$ consists of a compact action set A, a continuous utility function $u : A \times \Omega \to \mathbb{R}$, and a signal realization $s \in S$. The interpretation is that we are considering an agent with action set A and utility function u who has observed signal realization s. Let $\langle \pi | s \rangle$ denote the distribution of posteriors induced by observing signal π after having previously observed signal realization s: letting $Pr(\hat{s}) \equiv \sum_{\omega \in \Omega} p^{\omega}(\hat{s}) \mu_0^{\omega}$ denote the unconditional probability of realization $\hat{s}$ for any $\hat{s} \in S$, distribution $\langle \pi | s \rangle$ assigns probability $\sum_{\{s' \in \pi : \mu_{s \cap s'} = \mu\}} \frac{Pr(s \cap s')}{Pr(s)}$ to each belief μ. The value of a signal π in an interim decision problem $\hat{D}^i$ is given by $\mathbb{E}_{\tilde{\mu} \sim \langle \pi | s \rangle} \left[\max_{a \in A} \mathbb{E}_{\omega \sim \tilde{\mu}} u(a, \omega) \right]$. We say π *strongly Blackwell dominates* π' *in the interim sense* and write $\pi \overline{\mathcal{B}}^i \pi'$ if π has a higher value than π' for every $\hat{D}^i$.

A costly acquisition decision problem $\hat{D}^k = (A, u, c)$ consists of a compact action set A, a continuous utility function $u : A \times \Omega \to \mathbb{R}$, and a cost function $c : \Pi \to \mathbb{R}_+$ with $c(\underline{\pi}) = 0$. The interpretation is that we are considering an agent with action set A and utility function u who can, in addition to the signal whose value we are considering, acquire any additional signal $\hat{\pi}$ at cost $c(\hat{\pi})$.[30] The value of signal π in a costly acquisition decision problem is

$$\sup_{\hat{\pi} \in \Pi} \mathbb{E}_{\tilde{\mu} \sim \langle \pi \vee \hat{\pi} \rangle} \left[\max_{a \in A} \mathbb{E}_{\omega \sim \tilde{\mu}} u(a, \omega) \right] - c(\hat{\pi}).$$

We say π *strongly Blackwell dominates* π' *under costly information acquisition* and write $\pi \overline{\mathcal{B}}^k \pi'$ if π has a higher value than π' for every $\hat{D}^k$.

These two alternative notions are equivalent to strong Blackwell:

Proposition 1. $\overline{\mathcal{B}} = \overline{\mathcal{B}}^i = \overline{\mathcal{B}}^k$.

Proof. We first show that $\overline{\mathcal{B}} = \overline{\mathcal{B}}^i$. Suppose $\pi \overline{\mathcal{B}} \pi'$. Consider some interim decision problem (A, u, s). Let $\hat{\pi}$ be a signal that consists of s and, for each state, a signal realization (disjoint with s) that reveals that state, i.e., $\hat{\pi} = \{s\} \cup \{(\{\omega\} \times [0,1]) \setminus s | \omega \in \Omega\}$. Since $\pi \overline{\mathcal{B}} \pi'$, we know

[30] We could also consider the possibility that the agent chooses what additional costly information to acquire only after they observe the realization of the signal whose value we are considering. Once again, this notion would be equivalent to the strong Blackwell order.

$\mathbb{E}_{\tilde{\mu}\sim\langle\pi\vee\hat{\pi}\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right]\geq\mathbb{E}_{\tilde{\mu}\sim\langle\pi'\vee\hat{\pi}\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right]$, i.e.,

$$\sum_{\hat{s}\in\hat{\pi}}Pr\left(\hat{s}\right)\mathbb{E}_{\tilde{\mu}\sim\langle\pi|\hat{s}\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right]\geq\sum_{\hat{s}\in\hat{\pi}}Pr\left(\hat{s}\right)\mathbb{E}_{\tilde{\mu}\sim\langle\pi'|\hat{s}\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right].$$

For any $\hat{s}\in\hat{\pi}$ with $\hat{s}\neq s$, we have $\mathbb{E}_{\tilde{\mu}\sim\langle\pi|\hat{s}\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right]=\mathbb{E}_{\tilde{\mu}\sim\langle\pi'|\hat{s}\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right]$ since $\hat{s}$ fully reveals the state. Thus, we must have

$$\mathbb{E}_{\tilde{\mu}\sim\langle\pi|s\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right]\geq\mathbb{E}_{\tilde{\mu}\sim\langle\pi'|s\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right].$$

Since the choice of (A,u,s) was arbitrary, we conclude $\pi\overline{\mathcal{B}}^{i}\pi'$. Thus, $\overline{\mathcal{B}}\subseteq\overline{\mathcal{B}}^{i}$.

Conversely, suppose $\pi\overline{\mathcal{B}}^{i}\pi'$. Consider some extended decision problem $(A,u,\hat{\pi})$. We have that

$$\begin{aligned}\mathbb{E}_{\tilde{\mu}\sim\langle\pi\vee\hat{\pi}\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right]-\mathbb{E}_{\tilde{\mu}\sim\langle\pi'\vee\hat{\pi}\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right]&=\\ \sum_{\hat{s}\in\hat{\pi}}Pr\left(\hat{s}\right)\mathbb{E}_{\tilde{\mu}\sim\langle\pi|\hat{s}\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right]-\sum_{\hat{s}\in\hat{\pi}}Pr\left(\hat{s}\right)\mathbb{E}_{\tilde{\mu}\sim\langle\pi'|\hat{s}\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right]&=\\ \sum_{\hat{s}\in\hat{\pi}}Pr\left(\hat{s}\right)\left(\mathbb{E}_{\tilde{\mu}\sim\langle\pi|\hat{s}\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right]-\mathbb{E}_{\tilde{\mu}\sim\langle\pi'|\hat{s}\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right]\right)&.\end{aligned}$$

Since $\pi\overline{\mathcal{B}}^{i}\pi'$, we know $\mathbb{E}_{\tilde{\mu}\sim\langle\pi|\hat{s}\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right]-\mathbb{E}_{\tilde{\mu}\sim\langle\pi'|\hat{s}\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right]\geq 0$ for each $\hat{s}$ and thus $\mathbb{E}_{\tilde{\mu}\sim\langle\pi\vee\hat{\pi}\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right]\geq\mathbb{E}_{\tilde{\mu}\sim\langle\pi'\vee\hat{\pi}\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right]$. Since the choice of $(A,u,\hat{\pi})$ was arbitrary, we conclude $\pi\overline{\mathcal{B}}\pi'$. Thus, $\overline{\mathcal{B}}^{i}\subseteq\overline{\mathcal{B}}$.

We now show that $\overline{\mathcal{B}}=\overline{\mathcal{B}}^{k}$. Suppose $\pi\overline{\mathcal{B}}\pi'$. Suppose the value of π' on some costly acquisition decision problem (A,u,c) is v. For any $\epsilon>0$, let π^{ϵ} be any signal such that $\mathbb{E}_{\tilde{\mu}\sim\langle\pi'\vee\pi^{\epsilon}\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right]-c\left(\pi^{\epsilon}\right)\geq v-\epsilon$. Since $\pi\overline{\mathcal{B}}\pi'$, the value of π in (A,u,π^{ϵ}) is at least the value of π' in (A,u,π^{ϵ}), so that

$$\mathbb{E}_{\tilde{\mu}\sim\langle\pi\vee\pi^{\epsilon}\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right]\geq\mathbb{E}_{\tilde{\mu}\sim\langle\pi'\vee\pi^{\epsilon}\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right].$$

This implies that the value of π in the costly acquisition decision problem (A,u,c) is at least v, since

$$\sup_{\hat{\pi}\in\Pi}\mathbb{E}_{\tilde{\mu}\sim\langle\pi\vee\hat{\pi}\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right]-c\left(\hat{\pi}\right)\geq\mathbb{E}_{\tilde{\mu}\sim\langle\pi\vee\pi^{\epsilon}\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right]-c\left(\pi^{\epsilon}\right)$$
$$\geq\mathbb{E}_{\tilde{\mu}\sim\langle\pi'\vee\pi^{\epsilon}\rangle}\left[\max_{a\in A}\mathbb{E}_{\omega\sim\tilde{\mu}}u\left(a,\omega\right)\right]-c\left(\pi^{\epsilon}\right)$$
$$\geq v-\epsilon.$$

Since ϵ was arbitrary, we conclude that the value of (A, u, c) under π is greater than that under π'. Thus, π is more valuable than π' in every costly acquisition decision problem, and hence $\overline{\mathcal{B}} \subseteq \overline{\mathcal{B}}^k$.

Finally, suppose $\pi\overline{\mathcal{B}}^k\pi'$. Consider some extended decision problem $(A, u, \hat{\pi})$. Let

$$K=\max_{\omega\in\Omega}\left(\max_{a\in A}u\left(a,\omega\right)-\min_{a\in A}u\left(a,\omega\right)\right).$$

Clearly, the value of any signal (relative to $\underline{\pi}$) in the decision problem (A, u) is less than K. Let $c^* : \Pi \to \mathbb{R}$ be as follows: $c^*(\hat{\pi}) = c^*(\underline{\pi}) = 0$ and $c^*(\pi) = K$ for all $\pi \notin \{\hat{\pi},\ \underline{\pi}\ \}$. Then, the value of any signal in the extended decision problem $(A, u, \hat{\pi})$ must be the same as the value of that signal in the costly acquisition decision problem (A, u, c^*). Since $\pi\overline{\mathcal{B}}^k\pi'$, we know that π is more valuable than π' in (A, u, c^*); thus π is more valuable than π' in $(A, u, \hat{\pi})$. Since the choice of $(A, u, \hat{\pi})$ was arbitrary, we conclude $\pi\overline{\mathcal{B}}\pi'$. Thus, $\overline{\mathcal{B}}^k \subseteq \overline{\mathcal{B}}$. □

A.2 Characterization of the martingale relation

In Remark 4.6, we referred to the following observation, which we now state formally:

Proposition 2. *$\pi\mathcal{M}\pi'$ if and only if $C(\pi)\mathcal{S}\pi'$.*

As mentioned earlier, it is straightforward that $C(\pi)\mathcal{S}\pi'$ implies that $\pi\mathcal{M}\pi'$. Here, we provide the proof of the other direction, that $\pi\mathcal{M}\pi'$ implies $C(\pi)\mathcal{S}\pi'$.

We begin with some notation. Let $\mu_u \in \Delta(\Omega)$ indicate the uniform prior over states. Let $\Delta^o(\Omega)$ indicate the interior of the set of beliefs. Let $N = |\Omega|$. For a signal realization s, let $p(s)$ be the vector of probabilities of the signal realization s, i.e., $p(s) = (p^\omega(s))_{\omega\in\Omega}$. Denote its relative likelihood vector as $l(s) \equiv p(s)/\sum_\omega p^\omega(s)$. The relative likelihood vector is exactly the induced posterior from observing s under the uniform prior μ_u, and for future reference, we observe that

$l\,(s) \cdot \mu_u = 1/N$. Notice that, fixing an interior prior $\mu_0 \in \Delta^o(\Omega)$, the posterior belief after observing s is generated by a one-to-one mapping from $l(s)$ into $\Delta(\Omega)$. Hence, $C\,(\pi)$ pools together all of the realizations $s \in \pi$ that have identical relative likelihood vectors $l(s)$.

Now suppose that $C(\pi)$ is not sufficient for π'. We seek to show that π does not martingale dominate π'.

Because $\neg\,(C(\pi)\mathcal{S}\pi')$, there exist $\underline{s} \in \pi'$ and $\overline{s} \in C(\pi)$ that have a non-trivial intersection (i.e., $p^\omega(\underline{s} \cap \overline{s}) > 0$ for some ω) and $l(\overline{s}) \neq l(\underline{s} \cap \overline{s})$, since $l(\overline{s}) \neq l(\underline{s} \cap \overline{s})$ implies that posterior beliefs are different after observing $\overline{s}$ versus $\overline{s}$ and $\underline{s}$. Fix this element $\underline{s} \in \pi'$. Denumerate the elements of $C(\pi)$ that non-trivially intersect $\underline{s}$ as $\{\overline{s}_i\}_{i \in Q}$, and for each $i \in Q$ define $\underline{s}_i = \overline{s}_i \cap \underline{s}$. Observe that for $i \neq j$ in Q, we have that $l(\overline{s}_i) \neq l(\overline{s}_j)$, because any two signal realizations in $C(\pi)$ have different relative likelihood vectors. Note that there is some $i \in Q$ for which $l(\overline{s}_i) \neq l(\underline{s}_i)$; let $Q' \subseteq Q$ be the (non-empty) set of indices i at which $l(\overline{s}_i) \neq l(\underline{s}_i)$.

Claim 1. If there exists $\mu_0 \in \Delta^o(\Omega)$ such that

$$\left(\sum_{\omega \in \Omega} \sum_{i \in Q} p^\omega(\overline{s}_i) \frac{\sum_{\omega'} \mu_0^{\omega'} p^{\omega'}(\underline{s}_i)}{\sum_{\omega'} \mu_0^{\omega'} p^{\omega'}(\overline{s}_i)} \right) \neq \sum_{\omega \in \Omega} p^\omega(\underline{s}), \tag{1}$$

then $\neg\,(\pi \mathcal{M} \pi')$.

To prove the claim, first observe that $C\,(\pi)\,\mathcal{M}\pi'$ if and only if for all $\overline{s} \in C\,(\pi)$, $\underline{s} \in \pi'$, we have $\mathbb{E}\left[\tilde{\mu}_{C(\pi)} | \underline{s}\right] = \mu_{\underline{s}}$, i.e., for all ω:

$$\sum_{i \in Q} \sum_{\omega' \in \Omega} \underbrace{\frac{\mu_0^{\omega'} p^{\omega'}\,(\underline{s})}{\sum_{\omega''} \mu_0^{\omega''} p^{\omega''}\,(\underline{s})}}_{=\mu_{\underline{s}}^{\omega'} = Pr(\omega' | \underline{s})} \underbrace{\frac{p^{\omega'}\,(\underline{s}_i)}{p^{\omega'}\,(\underline{s})}}_{=Pr(\overline{s}_i | \underline{s}, \omega')} \underbrace{\frac{\mu_0^{\omega} p^{\omega}\,(\overline{s}_i)}{\sum_{\omega''} \mu_0^{\omega''} p^{\omega''}\,(\overline{s}_i)}}_{=\mu_{\overline{s}_i}^{\omega}} - \frac{\mu_0^{\omega} p^{\omega}\,(\underline{s})}{\sum_{\omega''} \mu_0^{\omega''} p^{\omega''}\,(\underline{s})} = 0$$

$$\iff \frac{\mu_0^\omega}{\sum_{\omega''} \mu_0^{\omega''} p^{\omega''}\,(\underline{s})} \left(\sum_{i \in Q} p^\omega\,(\overline{s}_i)\, \frac{\sum_{\omega'} \mu_0^{\omega'} p^{\omega'}\,(\underline{s}_i)}{\sum_{\omega''} \mu_0^{\omega''} p^{\omega''}\,(\overline{s}_i)} - p^\omega\,(\underline{s}) \right) = 0.$$

This expression holds for all $\mu_0 \in \Delta^o\,(\Omega)$ if and only if the term in parentheses is zero for all ω. Summing across ω gives the result.

Importantly, the RHS of (1) does not depend on μ_0. So we can guarantee that there exists an interior prior μ_0 at which the two sides are not equal as long as the LHS is not constant in μ_0.

Rewriting sums as dot products and simplifying further, we get the following implication.

Claim 2. Let $H_i(\mu) : [0,1]^N \to \mathbb{R}$ be defined as

$$H_i(\mu) \equiv \left(\sum_{\omega \in \Omega} p^\omega(\underline{s}_i) \right) \frac{\mu \cdot l(\underline{s}_i)}{\mu \cdot l(\overline{s}_i)}. \tag{2}$$

Let $H(\mu) \equiv \sum_{i \in Q'} H_i(\mu)$. If $H(\mu)$ is non-constant over the domain $\mu \in \Delta^o(\Omega)$, then $\neg(\pi \mathcal{M} \pi')$.[31]

We can consider two exhaustive and mutually exclusive cases.

Case 1: Q' is a singleton, which can be written as $Q' = \{i'\}$; and $l(\overline{s}_{i'}) = \mu_u$. In this case, $\mu \cdot l(\overline{s}_{i'}) = 1/N$ for all $\mu \in \Delta^o(\Omega)$, and hence $H(\mu) = N \left(\sum_\omega p^\omega(\underline{s}_{i'})\right) \mu \cdot l(\underline{s}_{i'})$. Moreover, because $l(\underline{s}_{i'}) \neq l(\overline{s}_{i'}) = \mu_u$, it also holds that $\mu \cdot l(\underline{s}_{i'})$ is linear and non-constant in μ. Hence, $H(\mu)$ is non-constant over the domain $\mu \in \Delta^o(\Omega)$.

Case 2: There exists some $i \in Q'$ such that $l(\overline{s}_i) \neq \mu_u$. We will find a direction $d_* \in \mathbb{R}^N$ with $\sum_\omega d_*^\omega = 0$ such that $H(\mu_u + \delta d_*)$ is nonconstant in δ in the neighborhood of $\delta = 0$, which will complete the proof.

Let

$$\hat{i} \in \underset{\{i \in Q' | l(\overline{s}_i) \neq l(\underline{s}_i)\}}{\arg\max} \|l(\overline{s}_i) - \mu_u\| \tag{3}$$

where $\|\cdot\|$ denotes the Euclidean norm, and set $d = \mu_u - l(\overline{s}_{\hat{i}})$. From the definition of $\hat{i}$, and the fact that $\mu \cdot \mu_u = l(s) \cdot \mu_u = 1/N$ for all s and $\mu \in \Delta(\Omega)$, we have

$$\begin{aligned}
(\mu_u + \delta d) \cdot l(\overline{s}_i) &= \mu_u \cdot l(\overline{s}_i) + \delta \left(\mu_u - l\left(\overline{s}_{\hat{i}}\right)\right) \cdot l(\overline{s}_i) \\
&= 1/N - \delta (l(\overline{s}_{\hat{i}}) - \mu_u) \cdot (l(\overline{s}_i) - \mu_u) \\
&\geq 1/N - \delta \|l(\overline{s}_{\hat{i}}) - \mu_u\| \|l(\overline{s}_i) - \mu_u\| \\
&\geq 1/N - \delta \|l(\overline{s}_{\hat{i}}) - \mu_u\|^2 \\
&= (\mu_u + \delta d) \cdot l(\overline{s}_{\hat{i}}).
\end{aligned}$$

(Note that the inequality is strict if $i \neq \hat{i}$, because then $l(\overline{s}_i) \neq l(\overline{s}_{\hat{i}})$.) Now take δ^* to be the unique

[31] The function $H_i(\mu)$ may be undefined at points μ that lead to a 0 denominator, but H_i (and therefore H) is defined everywhere on $\Delta^o(\Omega)$.

δ such that this last expression is equal to zero, i.e., $\delta^* \equiv 1/(N\|l(\overline{s}_{\hat{i}}) - \mu_u\|^2)$, and for all $i \neq \hat{i}$, we have that $(\mu_u + \delta^* d) \cdot l(\overline{s}_i) > 0$.

If in addition we have

$$(\mu_u + \delta^*(\mu_u - l(\overline{s}_{\hat{i}}))) \cdot l(\underline{s}_{\hat{i}}) \neq 0,$$

then take $d^* = d$. Otherwise, let d' be the projection of $l(\underline{s}_{\hat{i}}) - \mu_u$ onto the null space of $l(\overline{s}_{\hat{i}}) - \mu_u$, and note that $d' \neq 0$ because $l(\underline{s}_{\hat{i}}) \neq l(\overline{s}_{\hat{i}})$ (per (3)). Hence, for ϵ sufficiently small,

$$(\mu_u + \delta^*(d + \epsilon d')) \cdot l(\overline{s}_i) > 0 \ \forall i \neq \hat{i};$$
$$(\mu_u + \delta^*(d + \epsilon d')) \cdot l(\underline{s}_{\hat{i}}) \neq 0;$$
$$(\mu_u + \delta^*(d + \epsilon d')) \cdot l(\overline{s}_{\hat{i}}) = 0.$$

We then set $d^* = d + \epsilon d'$.[32]

Again using the fact that $\mu_u \cdot \mu = 1/N$ for any μ with $\sum_{\omega\in\Omega} \mu(\omega) = 1$, we have that for all $\delta \in [0, \delta^*)$ and for all i, $(\mu_u + \delta d^*) \cdot l(\overline{s}_i) > 0$. Hence, $H(\mu_u + \delta d^*)$ is finite for all $\delta \in [0, \delta^*)$, since the denominators of (2) are nonzero for every $i \in Q'$; and because at δ^* the numerator at $\hat{i}$ is non-zero, the denominator at $\hat{i}$ is zero, and the denominators at $i \neq \hat{i}$ are all non-zero, we have that

$$\lim_{\delta \nearrow \delta^*} H(\mu_u + \delta d^*) = \pm\infty.$$

Finally, note that $H(\mu_u + \delta d^*)$ is a rational function of δ (and therefore analytic in δ), is defined for all $\delta \in [0, \delta^*]$, and only has a singularity at $\delta = \delta^*$. Thus, H must be non-constant in δ on every open set in the interval $[0, \delta^*]$, and in particular, it is non-constant in the neighborhood of $\delta = 0$.

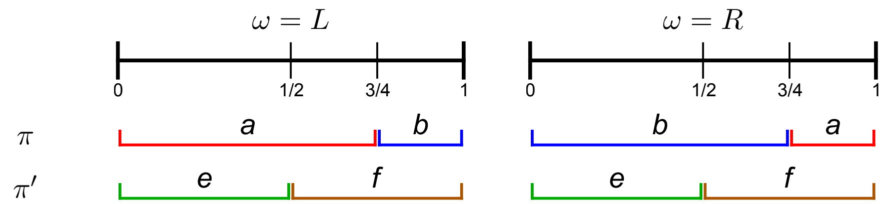
A.3 The martingale property can depend on priors

We define the martingale relation $\mathcal{M}$ as follows: $\pi \mathcal{M} \pi'$ if $\mathbb{E}[\tilde{\mu}_\pi | \tilde{s}_{\pi'}] = \tilde{\mu}_{\pi'}$ holds for all priors μ_0. In this section, we note that there are signals for which $\mathbb{E}[\tilde{\mu}_\pi | \tilde{s}_{\pi'}] = \tilde{\mu}_{\pi'}$ holds at some interior priors but not others. See Figure 7 for an example. Hence, the for-all quantifier on the prior has content.

In Figure 7, π is informative about the state, while π' is informative about how informative π is.

[32] Note that $\mu_u + \delta^* d^*$ need not be a probability vector in $\Delta(\Omega)$; the sum of components is 1, but it may have negative components. The rest of the proof shows that H is non-constant on the restricted domain of $\Delta^o(\Omega)$.

Figure 7: The martingale property can depend on priors



At prior μ_0 on $Pr(\omega = R)$, it holds that $\mu_a = \frac{\mu_0}{3-2\mu_0}$; $\mu_b = \frac{3\mu_0}{2\mu_0+1}$; $\mu_e = \mu_0$; and $\mu_f = \mu_0$. Moreover, $\mathbb{E}[\tilde{\mu}_\pi|e] = \mu_0\mu_a + (1-\mu_0)\mu_b = \frac{\mu_0(8\mu_0^2-14\mu_0+9)}{-4\mu_0^2+4\mu_0+3}$ and $\mathbb{E}[\tilde{\mu}_\pi|f] = \frac{1}{2}\mu_a + \frac{1}{2}\mu_b = \frac{\mu_0(5-2\mu_0)}{-4\mu_0^2+4\mu_0+3}$. It is easy to verify that $\mathbb{E}[\tilde{\mu}_\pi|e] = \mu_e$ and $\mathbb{E}[\tilde{\mu}_\pi|f] = \mu_f$ if $\mu_0 = 1/2$, and that these equalities do not hold at any other interior prior $\mu_0 \in (0,1)$.

In particular, signal π realizes either a, indicating a higher chance that $\omega = L$ and yielding $\mu_a < \mu_0$ (with beliefs in $[0,1]$ denoting the probability of $\omega = R$); or b, indicating a higher chance that $\omega = R$ and yielding $\mu_b > \mu_0$. When $e \in \pi'$ is realized, π is in fact perfectly informative: $a \in \pi$ implies that $\omega = L$ for sure, and $b \in \pi$ implies $\omega = R$. And when $f \in \pi'$ is realized, π is uninformative: a and b both have the same conditional likelihood across states. Since π' is uninformative about the state itself, though, the posterior after observing either realization from π' is always equal to the prior: $\mu_e = \mu_f = \mu_0$. But the expectation of $\tilde{\mu}_\pi$ (the posterior of π) given either realization of π' is equal to the prior only when beliefs are degenerate, or when the prior is uniform at $\mu_0 = 1/2$. This is easiest to see by considering $\mathbb{E}[\tilde{\mu}_\pi|f]$. Conditional on $f \in \pi'$, the signal π realizes a and b with equal probability, independently of the prior; but the posterior beliefs μ_a and μ_b are not equally distant from the prior. For instance, at priors $\mu_0 \in (0, 1/2)$, it holds that $\mu_b - \mu_0 > \mu_0 - \mu_a$.

Given that this martingale property can depend on the prior, we see that there is an alternative "martingale relation" on signals that we could have defined. Define the existence-martingale relation, denoted $\mathcal{M}^\exists$, as follows: $\pi\mathcal{M}^\exists\pi'$ if there exists an interior prior μ_0 at which $\mathbb{E}[\tilde{\mu}_\pi|\tilde{s}_{\pi'}] = \tilde{\mu}_{\pi'}$. It is easy to see that $\mathcal{M} \subsetneq \mathcal{M}^\exists$: the fact that $\mathcal{M} \subseteq \mathcal{M}^\exists$ follows immediately from definitions (for-all implies there-exists), and $\mathcal{M} \neq \mathcal{M}^\exists$ follows from the example in Figure 7. Moreover, it turns out that $\mathcal{M}^\exists \subsetneq \mathcal{B}$. The fact that $\mathcal{M}^\exists \subseteq \mathcal{B}$ can be seen by noting that if $\pi\mathcal{M}^\exists\pi'$, then for an interior prior μ_0 at which $\mathbb{E}[\tilde{\mu}_\pi|\tilde{s}_{\pi'}] = \tilde{\mu}_{\pi'}$, it holds that $\langle\pi\rangle$ is a mean-preserving spread of $\langle\pi'\rangle$; and if the posteriors of π are a mean-preserving spread of those of π' at any one interior prior, then they are a mean-preserving spread at all priors, i.e., $\pi\mathcal{B}\pi'$. The fact that $\mathcal{M}^\exists \neq \mathcal{B}$ can be established by observing that, in Figure 6, $\pi_{\mathcal{B}}\mathcal{B}\pi_0$, but, as can be directly calculated, $\neg(\pi_{\mathcal{B}}\mathcal{M}^{\exists}\pi_0)$.[33] Hence, we can expand our summary of the ranking of the relations to $\overline{\mathcal{R}} = \mathcal{R} \subsetneq \overline{\mathcal{S}} = \overline{\mathcal{M}} = \overline{\mathcal{B}} \subsetneq \mathcal{S} \subsetneq \mathcal{M} \subsetneq \mathcal{M}^{\exists} \subsetneq \mathcal{B}$.

A.4 An alternative characterization of $\mathcal{S}$, $\mathcal{M}$, and $\mathcal{B}$

Remark 4.7 stated the following characterizations of $\mathcal{S}$, $\mathcal{M}$, and $\mathcal{B}$.

1. $\pi\mathcal{S}\pi'$ if and only if $\exists\pi^*$ s.t. $C(\pi^*) = C(\pi)$, $\pi^*\mathcal{R}\pi$, and $\pi^*\mathcal{R}\pi'$. (We can take $\pi^* = \pi \vee \pi'$.)

2. $\pi\mathcal{M}\pi'$ if and only if $\exists\pi^*$ s.t. $C(\pi^*) = C(\pi)$ and $\pi^*\mathcal{R}\pi'$.

3. $\pi\mathcal{B}\pi'$ if and only if $\exists\pi^*$ s.t. $\pi^* \sim \pi$ and $\pi^*\mathcal{R}\pi'$.

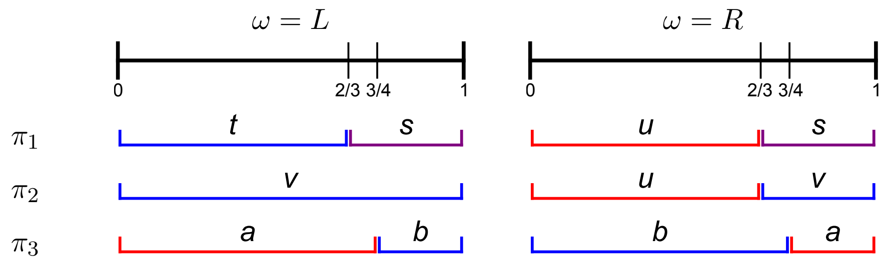
Part 3 had already been presented in Remark 4.2.

To see the only if direction of Part 1, first suppose that $\pi S \pi'$, and let $\pi^* = \pi \vee \pi'$. We see that $C(\pi^*) = C(\pi \vee \pi') = C(\pi)$, with the second equality following from the fact that $\pi\mathcal{S}\pi' \Leftrightarrow \tilde{\mu}_{\pi\vee\pi'} = \tilde{\mu}_{\pi}$. And by construction, $\pi^*\mathcal{R}\pi$ and $\pi^*\mathcal{R}\pi'$. Next, consider the if direction. Suppose that $\exists\pi^*$ s.t. $C(\pi^*) = C(\pi)$, $\pi^*\mathcal{R}\pi$, and $\pi^*\mathcal{R}\pi'$. The latter two properties imply that $\pi^*\mathcal{R}(\pi \vee \pi')$. Therefore, $C(\pi^*)\mathcal{B}C(\pi \vee \pi')\mathcal{B}C(\pi)$. This fact, coupled with $C(\pi) = C(\pi^*)$ implies that $C(\pi^*) \sim C(\pi \vee \pi') \sim C(\pi)$. Finally, $C(\pi \vee \pi') \sim C(\pi)$ implies that $\pi\mathcal{S}\pi'$.

Part 2 follows from Part 1 combined with the observation (Remark 4.6, Proposition 2) that $\pi\mathcal{M}\pi'$ if and only if $C(\pi)\mathcal{S}\pi'$. First, we suppose that $\pi\mathcal{M}\pi'$, and show that $\exists\pi^*$ s.t. $C(\pi^*) = C(\pi)$ and $\pi^*\mathcal{R}\pi'$. This holds because $\pi\mathcal{M}\pi'$ implies $C(\pi)\mathcal{S}\pi'$, which implies by Part 1 that there exists π^* (including $\pi^* = \pi \vee \pi'$) satisfying these conditions. Next, we suppose that $\exists\pi^*$ s.t. $C(\pi^*) = C(\pi)$ and $\pi^*\mathcal{R}\pi'$, and show that $\pi\mathcal{M}\pi'$. Take such a π^*, and observe that it satisfies $\pi^*\mathcal{R}C(\pi^*)$ and $C(\pi^*) = C(\pi)$, and hence $\pi^*\mathcal{R}C(\pi)$. Because π^* by definition also satisfies $C(\pi^*) = C(C(\pi))$ (since $C(C(\pi)) = C(\pi)$) and $\pi^*\mathcal{R}\pi'$, Part 1 implies that $C(\pi)\mathcal{S}\pi'$, which then implies $\pi\mathcal{M}\pi'$.

[33] Writing all beliefs in terms of $Pr(\omega = R)$, we have $\mu_k = \frac{\mu_0 \cdot 1/3}{\mu_0 \cdot 1/3 + (1-\mu_0)\cdot 2/3}$ at $k \in \pi_0$, along with $\mu_a = \frac{\mu_0 \cdot 1/4}{\mu_0 \cdot 1/4 + (1-\mu_0)\cdot 3/4}$ and $\mu_b = \frac{\mu_0 \cdot 3/4}{\mu_0 \cdot 3/4 + (1-\mu_0)\cdot 1/4}$ at a and b in $\pi_{\mathcal{B}}$. Conditional on realization $k \in \pi_0$, the expected belief at $\pi_{\mathcal{B}}$ is given by $\mathbb{E}[\tilde{\mu}_{\pi_{\mathcal{B}}}|k] = (1-\mu_k)\mu_a + \mu_k(\mu_a \cdot 3/4 + \mu_b \cdot 1/4)$. The martingale property at prior μ_0 holds only if $\mathbb{E}[\tilde{\mu}_{\pi_{\mathcal{B}}}|k] - \mu_k = 0$, but the LHS simplifies to $-\frac{\mu_0(1-\mu_0)}{6+5\mu_0-12\mu_0^2+4\mu_0^3}$, which has no zeroes for $\mu_0 \in (0,1)$.

Figure 8: Martingale is not transitive



It holds that $\pi_1 \mathcal{M} \pi_2$ and $\pi_2 \mathcal{M} \pi_3$ but not that $\pi_1 \mathcal{M} \pi_3$.

A.5 The martingale relation is not transitive

Consider Figure 8. We see that $\pi_1 \mathcal{M} \pi_2$ because $\pi_1 \mathcal{R} \pi_2$; and $\pi_2 \mathcal{M} \pi_3$ because $\pi_2 \mathcal{S} \pi_3$, which may not be immediately obvious.[34] However, it is not the case that $\pi_1 \mathcal{M} \pi_3$: with a prior of $\mu_0 = 1/2$ probability on $\omega = R$, we have that $\mu_a = 1/4$ while $\mathbb{E}[\tilde{\mu}_{\pi_1}|a] = \frac{2}{3} \cdot 0 + \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$.

A.6 Discriminating domains

Proposition 3. *If $\mathbb{D}$ is not discriminating, then $\overline{\mathcal{V}}_{\mathbb{D}} \neq \mathcal{O}$.*

Proof. Suppose that $\mathbb{D}$ is not discriminating. There exist distinct $\omega, \omega' \in \Omega$ such that for every decision problem in $\mathbb{D}$, $\arg\max_{a \in A} u(a, \omega) \cap \arg\max_{a \in A} u(a, \omega') \neq \emptyset$. Hence, there is an action that is optimal in both ω and ω'.

Now, consider the signals

$$\pi = \{\{\omega, \omega'\} \times [0,1], (\Omega \setminus \{\omega, \omega'\}) \times [0,1]\}$$

$$\pi' = \{\{\omega\} \times [0,1], \{\omega'\} \times [0,1], (\Omega \setminus \{\omega, \omega'\}) \times [0,1]\}.$$

Clearly, π does not reveal or refine π'. Yet, $\pi \mathcal{V}_{\mathbb{D}} \pi'$: the signals differ only in whether they distinguish ω from ω', but there is an action that is optimal in both states, so there is no value in distinguishing those states. □

[34] To see that $\pi_2 \mathcal{S} \pi_3$, or in other words that $\pi_2 \vee \pi_3$ induces the same beliefs as π_2, observe that $\pi_2 = \{u, v\}$ while $\pi_2 \vee \pi_3 = \{a \cap v = a, b \cap v, b \cap u = u\}$. So it suffices to show that the likelihood ratios of (and thus the beliefs at) a and of $b \cap v$ match that of v, which indeed they do: $Pr(v|L)/Pr(v|R) = Pr(b \cap v|L)/Pr(b \cap v|R) = Pr(a|L)/Pr(a|R) = 3$.